Goldstein

3.11. $$L = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + \frac{k}{r}$$

$\dot{\theta} \to 0$ because motion stopped.

$$\Rightarrow \quad L = \frac{1}{2}m(\dot{r}^2) + \frac{k}{r}.$$

$$\frac{dL}{dr} = -\frac{k}{r^2}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r},$$

Eqm: $$m\ddot{r} = -\frac{k}{r^2}.$$

$$\Rightarrow \quad m\ddot{r}\dot{r} = -\frac{k}{r^2}\dot{r},$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = \frac{d}{dt}\left(\frac{k}{r}\right)$$

$$\frac{1}{2}m\dot{r}^2 = \frac{k}{r} + c,$$

$$m\dot{r}^2 = \frac{2k}{r} + 2c.$$

$$\dot{r}^2 = \frac{2k}{mr} + \frac{2}{m}c.$$

We impose $\dot{r} = 0$ at $r = r_0$, then $c = -\frac{k}{r_0}$, and

$$\dot{r}^2 = \frac{2k}{mr} - \frac{2k}{mr_0},$$

$$\dot{r}^2 = \frac{2k}{m}\left[\frac{1}{r} - \frac{1}{r_0}\right].$$

$$\boxed{\dot{r} = \sqrt{\frac{2k}{m}}\sqrt{\frac{1}{r} - \frac{1}{r_0}}},$$

The diff. eq. above shall be put in a form that is easy to integrate,

$$\frac{1}{r} - \frac{1}{r_0} = \frac{r_0 - r}{r_0 r}$$

$$\dot{r} = \sqrt{\frac{2k}{m}} \sqrt{\frac{r_0 - r}{r_0 r}},$$

$$= \sqrt{\frac{2k}{m}} \sqrt{\frac{1 - r/r_0}{r}},$$

$$\Rightarrow \sqrt{\frac{m}{2k}} \sqrt{\frac{r}{1 - r/r_0}}\, dr = dt.$$

We are interested in the region where $0 \leq r \leq r_0$, so we can let $r = r_0 \sin^2\theta$, then we have

$$\sqrt{\frac{m}{2k}} \sqrt{\frac{r_0 \sin^2\theta}{1 - \sin^2\theta}}\, 2 r_0 \sin\theta \cos\theta \, d\theta = dt.$$

$$\sqrt{\frac{m}{2k}} (2r_0)(\sqrt{r_0}) \frac{\sin\theta}{\cos\theta} \sin\theta \cos\theta \, d\theta = dt.$$

$$2 \sqrt{\frac{m}{2k}}\, r_0^{3/2} \sin^2\theta \, d\theta = dt.$$

$$2 \sqrt{\frac{m}{2k}}\, r_0^{3/2} \int_{\pi/2}^{0} \sin^2\theta \, d\theta = \int dt.$$

The above integral can be evaluated simply by the fact that $\sin^2\theta$ has average value of $\frac{1}{2}$ over half wavelengths and is symmetric about $\pi/2$.

$$\Rightarrow \left| \int_{\pi/2}^{0} \sin^2\theta \, d\theta \right| = \frac{\pi}{2} \frac{1}{2} = \frac{\pi}{4}$$

$$\Rightarrow \Delta t = \frac{2\pi}{4} \sqrt{\frac{m}{2k}}\, r_0^{3/2} = \boxed{\frac{2\pi}{4\sqrt{2}} \sqrt{\frac{m r_0^3}{k}}}$$

off by a constant?

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